Does stock market investment matter for local indeterminacy?

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Abstract

The purpose of this paper is to study the influence of stock market investment on the local dynamics of a standard overlapping generations (OLG) economy with capital accumulation. In particular, we focus on the emergence of local bifurcations and expectations-driven fluctuations in an economy with endogenous profits. We consider an OLG model with future consumption, decreasing returns to scale in production, and investment in both physical capital and firms' shares. We analyze the local dynamics of our model in terms of relevant parameters, namely, the capital share of output, the elasticity of capital-labour substitution, and the degree of returns to scale. We conclude that the steady state is never a sink, and thus indeterminacy does not emerge (and hence does not hold as a possible explanation for the excess volatility puzzle at an aggregate level). The steady state is either a saddle or a source and undergoes a transcriptical bifurcation for values of the capital share of output sufficiently high. Finally, we compare steady state welfare with and without a secondary stock market. We conclude that the introduction of a secondary market for shares may correct the dynamic inefficiency that arises in a simple OLG model, improving steady state welfare when the capital share of output is sufficiently low (i.e. under strong dynamic inefficiency).

Keywords: Indeterminacy, local dynamics, secondary stock market, decreasing returns to scale, endogenous profits, dynamic inefficiency, steady state welfare.

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Part I
Introduction

Endogenous fluctuations, a widely studied topic in Macroeconomic and Dynamics and Control theories, accounts for a considerable share in what theorists have recently produced. Until the late 80's, macroeconomic theory was almost solely based on models of exogenous economic fluctuations, where equilibria is determinate and intrinsically stable. In spite of these models' explanation for business cycles relying strongly on the existence of continuing exogenous shocks to the economic fundamentals, these models are still the most popular amongst macroeconomics theorists and empiricists. This popularity derives, among other reasons, from the consensus around the fact that aggregate economy is affected by exogenous shocks to the economic fundamentals, and in particular to the random nature of these changes. A contrasting case is however proposed with "sunspot" equilibrium models. Under this approach, studying the set of circumstances (in terms of parameters of interest) that lead to a locally indeterminate steady state is quite relevant[2]. As stated in Woodford (1990), local indeterminacy creates the possibility of equilibrium fluctuations in response to events ("sunspots") that do not change economic fundamentals. Simultaneously, works under the endogenous fluctuations approach also focus on the occurrence of bifurcations, a feature strongly related to the emergence of both deterministic and stochastic fluctuations (see Grandmont et al. (1998)). In fact, even when the steady state is locally determinate, cycles may be present as long as bifurcations occur[3]. Note that the nonlinearities in the system may contain the dynamics (that otherwise would be explosive) in a neighborhood of the steady state. In brief, both indeterminacy and bifurcations are possible explanations to the endogenous fluctuations inherent to business cycles. Accordingly, the set of circumstances under which these features occur is worth to be studied in the context of macroeconomics models.

In the latter type of studies, however, indeterminacy has been shown to depend heavily on market failures. On such grounds, Lloyd-Braga, T., L. Modesto and T. Seegmuller (2011) found that capital market distortions do not seem able, per se, of influencing the dynamics of the model. This might be due to the fact that capital market features have not been sufficiently detailed. In particular, stock market exchanges are absent. Hence, it primarily becomes necessary to understand to what extent does the absence of stock market exchanges limit the results obtained in Lloyd-Braga et al. (2011). On this topic, and despite no macroeconomic model has, to my knowledge, addressed this question, several

[2]Local indeterminacy occurs when the number of eigenvalues lower than one in absolute value is higher than the number of predetermined variables. Therefore, indeterminacy implies a continuum of equilibrium trajectories, starting from the same value for the predetermined variable, that stay arbitrarily close to the steady state.

[3]A bifurcation is said to occur when one eigenvalue crosses the unit circle as some parameter of the model is made exogenously to vary within its admissible range of values.

empirical studies have pointed towards the need for a more comprehensive look at stock market exchanges.

First, Schiller (1981) reported that the volatility of the S&P 500 is far too high – 5 to 13 times too high – to be explained by new information about future dividends. This property has been named excess volatility of stock market returns, and is nowadays considered a stock market returns' stylized feature. On this property, several authors have come up with possible explanations, but none has so far reached any consensus. West (1988) stated that the conditions that make rational bubbles possible are too stringent, thus making this explanation particularly unattractive for excess volatility. Consequently, this should imply the existence of other factors (but rational bubbles) explaining the excess volatility implicit in stock market returns. Blanchard (1979) and Blanchard and Watson (1983) also introduced rational bubbles to explain excess volatility, but the set of circumstances that make this explanation plausible in a general equilibrium framework was found to be narrow. Regarding this discussion, indeterminacy may also be pointed out as a possible explanation for the excess volatility in the stock market. If indeterminacy holds true at an aggregate level, then endogenous fluctuations are known to emerge. Therefore, an additional source of volatility may be due to self-fulfilling expectations.

Second, Kose, M., E. Prasad and M. Terrones (2003) report two important results on financial integration. The volatility of consumption growth relative to that of income growth has increased for more financially integrated developing economies in the 1990s. Also, increasing financial openness is associated with rising relative volatility of consumption. These results suggest that financial integration and aggregate volatility go hand-in-hand. Schwert (1989) also shows that stock market volatility is higher on average during recessions. This fact suggests that stock market is a good business cycle indicator. More recently, Claessens, S., M. Kose and M. Terrones (2011) study the interactions between business and financial cycles using a database covering 44 countries for the period 1960:1-2010:4. Their study confirms the strong linkages between the different phases of business and financial cycles. In particular, they show that equity price busts tend to be associated with deeper and longer recessions. Hence, these empirical studies have proved a strong relation between equity markets and business cycles (in terms of volatility and cyclicality).

Bearing in mind the recent findings of these empirical studies, we are led to suspect that the absence of stock market exchanges restricts the possibility for local indeterminacy. Therefore, in this work we introduce a stock market and study its influence on the emergence of indeterminacy and bifurcations. Most of the models considered to evaluate the emergence of endogenous cycles assume that private technologies exhibit constant returns to scale, and thereby there are no profits to be distributed (see, e.g. Cazzavillan et al. (1998), Schmitt-Grohe (1997), Venditti et al. (2007)). The works considering market structures under which profits may arise usually specify exogenously the ownership of firms and who is entitled to receive profits. See, e.g. Seegmuller (2005) and Dos Santos Ferreira and Lloyd-Braga (2002), where an OLG economy with market imperfections is considered and profits are distributed to the young generation. In contrast, our paper considers an OLG economy under perfect competition with profits due to decreasing returns to scale and where exchanges of firms' shares take place in a secondary stock market: young workers save for consumption when old, choosing the amount of productive capital and firm's shares they wish to hold. In this set up profits end up being distributed to the old generation. Since we consider that the number of shares outstanding is fixed, our focus is on the secondary stock market. We analyze how these features influence the emergence of endogenous cycles.

We will consider a model where households only consume when old (as for instance in Reichlin (1986), Dos Santos Ferreira, R. and T. Lloyd-Braga (2005) and Coimbra, R., Lloyd-Braga, T. and L. Modesto (2005)) and where labor is supplied inelastically by young households: households save all their wage income received when young for future consumption. In this framework, if savings are all invested in productive capital (that will be rented to firms in the next period) there cannot be indeterminacy. The reason is that all endogenous variables become predetermined by past savings and thus cannot be influenced by expectations for the future. However, if agents can also choose to save in firms' shares, indeterminacy might, a priori, arise. In this case, capital accumulation today, that influences production tomorrow, will depend on current expectations about future returns of firm's shares. Therefore, local indeterminacy may emerge. Finally, we study how does the introduction of a secondary market for shares influence steady state welfare. As known, dynamic inefficiency[4] may arise in the simple overlapping generations model (see, e.g. Phelps (1961), Diamond (1965), Weil (1987), Tirole (1985)). With the introduction of a secondary market, firms' shares compete with productive capital in attracting the use of savings. Hence, it may correct for the over accumulation of capital and increase aggregate consumption at the steady state (and therefore steady state welfare). Accordingly, we study how does steady state welfare compare in an economy with and without a secondary stock market.

This paper is organized as follows: Part II sets up our model. Sections 1 and 2 describe respectively the problem faced by the representative consumer and firm. In Section 3 equilibrium is defined. Section 4 states the necessary conditions for the existence of a steady state. Section 5 presents the characteristic polynomial of our dynamic system. Section 6 uses the geometrical method developed in Grandmont et al. (1998) to study the local dynamics of our model. In section 7 we analyze steady state welfare in this economy and compare it to our benchmark (an economy with the same characteristics, but without a secondary stock market). Finally, Part III presents the concluding remarks, together with suggestions for further research. All the main proofs are gathered in a final Appendix.

[4]Dynamic inefficiency is said to occur when the steady state value for the capital stock exceeds the golden rule level. Hence, a decrease in the capital stock increases aggregate consumption in steady state. This is due to the fact that the over accumulation of capital makes its productivity insufficient to supply the resources.

Part II
The model

The model follows Devereux and Lockwood (1991). We consider a perfectly competitive economy with discrete time $t \in \{1, 2, ..., \infty\}$. In each period t, the economy is populated by two-period lived generations, the young of period t and the old of period $t-1$. The economy exhibits no population growth. The representative agent has preferences for future consumption according to the utility function $U(c)$, with c standing for future consumption. Each agent works while young, supplying $\bar{L}$ units of labour inelastically. There are N identical firms in the economy and n identical young households. In each period t there is a single final output in the economy, produced by each of the N firms out of capital and labour contracted under perfect competition. In addition, the final good is produced using a representative technology characterized by decreasing returns to scale and can be either used as capital or consumption good.

In contrast to the standard Devereux and Lockwood (1991) model[5], pure profits are generated through decreasing returns to scale at the firm level. Therefore, this economy exhibits endogenous profits in equilibrium, hence allowing a secondary stock market to emerge (there is no market for new shares) - the economy is initially endowed with $m > 0$ shares per firm. The young agents of each generation have thus to decide how much to invest in real assets, i.e. productive capital, and how much to invest in the purchase of firms' shares. In a nutshell, we may describe our economy as follows:

At the beginning of period t young agents receive their wage income and decide its allocation between investment in productive capital and investment in shares. In the next period, productive capital bought by young agents at time t is rented out to one of the N firms, and used in the production of the final output. Moreover, the old of period $t+1$ become the owners of the shares, and sell it to the young of period $t+1$ at the end of the period. Accordingly, at $t+1$ old agents receive the dividend income, the return on physical capital and the revenue from the shares sold. The total revenue stream is then used by the old of $t+1$ to consume. This process is rolled over forever.

The two sections below describe both the consumption and technological side of our model and state appropriate assumptions on the properties of the utility and production functions.

[5]In Devereux and Lockwood (1991) positive pure profits were held in equilibrium. However, positive profits emerged as result of frictions in the labour-wage negotiation. A stock market was then introduced as a tool to close the market, even tough it was not the purpose of their research.

1. Consumers

On the consumers side, there are overlapping-generations of two-period lived consumers. Each agent supplies $\bar{L}$ units of labour in the first period of life (receiving the wage income), saves through stock market investment and productive investment, and consumes in the future. Moreover, consumers have identical preferences for consumption according to $U(c)$. The properties of the utility function are specified below.

Assumption 1. *$U(c)$ is C^r over $\mathbb{R}_+$ for r large enough, increasing with respect to its argument ($U_c(c) > 0$) and concave ($U_{cc}(c) < 0$) over $\mathbb{R}^2_{++}$.*

Each agent maximizes its level of future consumption by transferring resources over time through investment in productive capital and the purchase of firms' shares. Hence, the representative agent maximizes its utility from future consumption subject to the two periods budget constraints. The representative consumer's problem may thus be represented as

$$
\begin{aligned}
&Max_{\{c_{t+1},k_{t+1},\theta_{t+1}\}} U(c_{t+1}) \\
&\qquad s.t. \quad k_{t+1} + p_t\theta_{t+1} = w_t\bar{L}, \\
&\qquad\qquad\; c_{t+1} = R_{t+1}k_{t+1} + (\tfrac{\Pi^N_{t+1}}{m} + p_{t+1})\theta_{t+1},
\end{aligned}
\tag{1}
$$

where w_t represents the wage income, θ_{t+1} are total purchases of shares per agent at time t, p_t represents the share price of a representative firm at time t, R_{t+1} is the real interest factor on productive capital and Π^N_{t+1} are nominal profits per firm. Note that the capital stock is here considered to be fully depreciated ($\delta = 1$), with δ standing for the depreciation rate[6]. Therefore, the real interest factor equals r_{t+1} (i.e. the real rental rate) in this economy. Accordingly, from this moment onwards our model will be specified in terms of r_{t+1}. From (1), we may interpret that the first budget constraint imposes that the representative consumer splits his wage income between investment in productive capital and investment in shares. Also, the second budget constraint sets that future consumption must be fully financed with income generated from savings, i.e. return earned on productive capital ($r_{t+1}k_{t+1}$), dividend income earned on the shares owned ($\frac{\Pi^N_{t+1}}{m}\theta_{t+1}$) and revenue from shares sold ($p_{t+1}\theta_{t+1}$). We focus on equilibria were both types of assets are held in equilibrium, and therefore the following arbitrage condition must bind

$$
r_{t+1} = \frac{\frac{\Pi^N_{t+1}}{m} + p_{t+1}}{p_t}. \tag{2}
$$

[6]Within the two-periods OLG models, full depreciation is justified by the fact that periods are sufficiently long, i.e. equivalent to one generation. On the contrary, within the infinite horizon models, periods are usually short and partial depreciation is often considered. We consider in our model full depreciation of productive capital. Therefore, the real interest factor equals the real rental rate, i.e. $R_{t+1} = 1 - \delta + r_{t+1} = r_{t+1}$.

The arbitrage condition in (2) together with the two budget constraints represented in (1) are the first order conditions of the representative consumer's problem.

2. Production

In each period $t = 1, 2, ..., \infty$, the final good is produced under a representative technology $AF(K_t, L_t)$, with $F(K, L)$ homogeneous of degree lower than one and $A > 0$ a scaling parameter. At the beginning of time t every firm is entirely owned by the older generation. Individuals' consumption when old depend on $p_t + \frac{\Pi_t^N}{m}$. Therefore, each firm maximizes $\frac{\Pi_t^N}{m} + p_t = \frac{\Pi_t^N}{m} + \sum_{\tau=1}^{\infty} D_\tau^{-1} \frac{\Pi_{t+\tau}^N}{m}$, with $D_\tau = \prod_{i=1}^{\tau}(1 + r_{t+i})$ and $i > 0$, i.e. the present value of future dividends[7]. Firm's objective function is hence separable and equivalent to the maximization of each period's dividends, with $\Pi_t^N = AF(K_t, L_t) - w_t L_t - r_t K_t$. Denoting, for $L \neq 0$, $x \equiv K/L$ the capital stock per unit of labour employed, the production function may be defined in intensive form as $AF(K_t, L_t) = AL_t^\gamma f(x_t)$, with $\gamma < 1$. The properties of the production function are characterized below.

Assumption 2. *$f(x)$ is C^r over $\mathbb{R}_+$ for r large enough, increasing with respect to each argument ($f^{'}(x) > 0$), concave ($f^{''}(x) < 0$) over $\mathbb{R}_{++}^2$ and homogeneous of degree lower than one. Moreover, it is further assumed that $\gamma f(x_t) - f^{'}(x_t)x_t > 0$.*

Profit maximization for a representative firm therefore implies that the real wage rate w_t and the real rental rate r_t are respectively equal to the marginal productivities of labour and capital:

$$w_t = AF_L(K_t, L_t) = AL_t^{\gamma-1}\left[\gamma f(x_t) - f^{'}(x_t)x_t\right], \tag{3}$$

$$r_t = AF_K(K_t, L_t) = A{L_t}^{\gamma-1} f^{'}(x_t). \tag{4}$$

Dividends may then be computed as the excess output revenue over labour and capital expenses. From (3) and (4), we obtain that dividends are given by

$$\Pi_t^N = AL_t^\gamma (1 - \gamma) f(x_t) > 0. \tag{5}$$

Therefore, for $\gamma < 1$ (decreasing returns to scale) pure profits endogenously emerge in this economy. This feature allows stock market returns to be directly linked to firms' profits. For future reference, we also computed the share of capital in total income[8].

[7]Under a scenario with no principal-agent problems, the firm's objective function is entirely aligned with its owners' objective function. Therefore, firms engage in the maximization of the present value of future dividends.

[8]Note that the share of capital in total income equals the capital share of output under competitive equilibrium. Moreover, the labour share of output may be represented as $\gamma - s(x)$.

$$s(x) = \frac{xf^{'}(x)}{f(x)} \in (0, \gamma), \tag{6}$$

and the elasticity of capital-labour substitution

$$\frac{1}{\sigma(x)} = \left\{ \frac{(\gamma - 1)}{\gamma} s - \frac{f^{''}(x)x}{f^{'}(x)} \right\} \frac{\gamma}{\gamma - s}. \tag{7}$$

Proof. See Appendix A. □

3. Equilibrium

First, for simplicity's sake we consider a fairly common assumption among OLG models - the number of individuals n is identical to the number of firms N. Then, equilibrium in the labour and capital markets implies that $\bar{L} = L_t$, $w_t = AF_L(K_t, \overline{L})$, $K_{t+1} = k_{t+1}$, $r_t = AF_K(K_t, \overline{L})$. Moreover, considering $m > 0$ as the constant number of shares outstanding, at the stock market equilibrium we have that $\theta_{t+1} = m$. Finally, equilibrium in the productive capital and shares markets sets that both the arbitrage condition and the budget constraints hold true. Therefore:

Definition 1. *A perfect foresight intertemporal equilibrium is a sequence* $(K_{t+1}, p_t) \in \mathbb{R}^2_{++}$, $t = 1, 2, ..., \infty$, *that, for a given* $K_0 > 0$, *satisfies:*

$$AF_L(K_t, \bar{L})\bar{L} - mp_t = K_{t+1} \tag{8}$$

$$AF_K(K_{t+1}, \bar{L})p_t = \frac{\Pi^N_{t+1}}{m} + p_{t+1} \tag{9}$$

where $w_t = AF_L(K_t, \bar{L}) = A\bar{L}^{\gamma-1} \left[\gamma f(x_t) - f^{'}(x_t)x_t \right]$, $r_{t+1} = AF_K(K_{t+1}, \bar{L}) =$

$= A\bar{L}^{\gamma-1} f^{'}(x_{t+1})$ *and* $\Pi^N_{t+1} = A(1-\gamma)F(K_{t+1}, \bar{L}) = A\bar{L}^{\gamma}(1-\gamma)f(x_{t+1})$.

We remark that, as represented in $(8)-(9)$, the dynamics of this economy are governed by a pair of two nonlinear difference equations in K_t, p_t. The capital stock is entirely determined by past actions, and hence said to be predetermined. The stock price, on the contrary, is affected by expectations about future events, and therefore non predetermined[9]. The intertemporal sequence of K_{t+1} and

[9]Note that although consumption is a predetermined variable (from (1) and (2), $c_{t+1} = AF_K(K_t, \overline{L})AF_{\overline{L}}(K_t, \overline{L})\overline{L})$ investment in period t, $K_{t+1} - K_t(1-\delta) = K_{t+1}$, depends on the decision about K_{t+1}. Then, K_{t+1} depends on the value for p_t, and accordingly on future expectations.

p_t enables us to determine all the other endogenous variables, namely c_{t+1}, $AF(K_{t+1}, \bar{L})$, r_{t+1}, w_t and $\frac{\Pi^N_{t+1}}{m}$. Finally, in order to study the local dynamics and analyze how does the introduction of stock market investment affect the local stability properties around the steady state, we first ensure the existence of a steady state. Then, the Hartman-Grobman Theorem applies, i.e. there is a neighbourhood of the steady state such that the equilibrium trajectories of the nonlinear system are similar to those of the linearized one (in terms of its qualitative properties). Accordingly, we secondly log-linearize the system $(8)-(9)$ around a neighborhood of the steady-state and analyze its local stability properties.

4. Steady State

In this section, we establish conditions for the existence of a steady state, i.e. a stationary solution $K_{t+1} = K_t = K$ and $p_{t+1} = p_t = p$, of the dynamic system in $(8) - (9)$. Given $\overline{L}$, a steady state satisfies:

$$A\bar{L}^{\gamma}\left[\gamma f(x) - f^{'}(x)x\right] - mp = K, \tag{10}$$

$$A\bar{L}^{\gamma-1}f^{'}(x) - \frac{A\bar{L}^{\gamma}(1-\gamma)f(x)}{pm} = 1. \tag{11}$$

We then use the scaling parameters A and m to describe necessary and sufficient conditions to ensure the existence of a given steady state (K, p).

Proposition 2. *Let* $\sigma_3^* \equiv \frac{1}{2-\gamma}$ *and* $m^0 \equiv \frac{(1-\gamma)\overline{L}^{\gamma} f(\overline{x})}{\overline{L}^{\gamma-1} f^{'}(\overline{x})\overline{p}}$. *Then,* (K, p) *is a steady state with* $x = {}^{K}/{}_{L}$ *if and only if*

$A = \left[\frac{\overline{x}+({}^{m\overline{p}}/{}_{\overline{L}})}{\gamma f(\overline{x})-f^{'}(\overline{x})\overline{x}}\right]\frac{1}{\overline{L}^{\gamma-1}}$ *and* $m > m^0$ *is the unique solution of*

$1 = Z(m) \equiv \left[\frac{\overline{x}+({}^{m\overline{p}}/{}_{\overline{L}})}{\gamma f(\overline{x})-f^{'}(\overline{x})\overline{x}}\right]\frac{1}{\overline{L}^{\gamma-1}}\left[\overline{L}^{\gamma-1}f^{'}(\overline{x}) - \frac{(1-\gamma)\overline{L}^{\gamma}f(\overline{x})}{m\overline{p}}\right].$

Moreover, for a technology exhibiting a constant elasticity of substitution between labour and capital, under $\sigma > \sigma_3^*$ *(and given* $m > m^0$ *and* A *as defined above)* $(\overline{K}, \overline{p})$ *is the unique steady state solution of* $(10) - (11)$.

Proof. *See Appendix B.*1 □

5. Characteristic polynomial

We start by log-linearizing the system of two non-homogeneous difference equations (8) − (9) around the steady state. Defining variables in terms of percent deviations around the steady state as $\hat{x} = \frac{x-\overline{x}}{\overline{x}}$, we have that

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{p}_{t+1} \end{bmatrix} = \begin{bmatrix} J \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{p}_t \end{bmatrix}, \tag{12}$$

with J representing the Jacobian matrix of the system $(8)-(9)$ evaluated at the steady state. The local stability properties of the model are thus determined by the eigenvalues of the Jacobian matrix J, i.e. the values λ_1 and λ_2 that solve $\mid J - \lambda I \mid = 0$ (with I equal to an identity matrix with the same dimension as J) . Equivalently, the local stability properties may be determined by the trace T and determinant D of the Jacobian matrix, which correspond respectively to the product and sum of the two eigenvalues of the characteristic polynomial $P(\lambda) = \lambda^2 - \lambda T + D$. We may hence state that $\lambda_1 = \frac{T-\sqrt{T^2-4D}}{2}$ and $\lambda_2 = \frac{T+\sqrt{T^2-4D}}{2}$, with λ_1 and λ_2 satisfying $D = \lambda_1\lambda_2$ and $T = \lambda_1 + \lambda_2$. Moreover, when $\mid \lambda \mid < 1$ the steady state is locally stable and said to be locally a sink. If $\mid \lambda_1 \mid < 1$ or $\mid \lambda_2 \mid < 1$ the steady state is locally a saddle. Finally, when $\mid \lambda \mid > 1$ the steady state is locally a source.

Proposition 3. *Under Assumptions 1-2, the characteristic polynomial is*

$$P(\lambda) = \lambda^2 - \lambda T + D$$

where the trace and determinant verify

$$T = \frac{s(1+\rho)}{\gamma(\gamma - s)} \left[\gamma + (1-\gamma)s(1+\rho)\right] + \frac{1}{\sigma}\left[\frac{s(1+\rho)^2}{\gamma}\right], \tag{13}$$

$$D = \frac{\left[(1+\rho)s(\gamma-1) + \frac{(1+\rho)s}{\sigma}\right]}{\gamma} \frac{s(1+\rho)}{(\gamma - s)}, \tag{14}$$

and $\rho \equiv \frac{mp}{k}$. *Moreover, from the arbitrage condition it must be that*

$$\rho = \frac{1 - 2s + \sqrt{(1-2s)^2 + 4s(1-\gamma)}}{2s}. \tag{15}$$

Proof. See Appendix B.2 and B.3 □

From $(13)-(14)$, we may compute the start and end point for T and D, i.e. the values of T and D at the boundary values of $\sigma \in [0, +\infty)$,

$$T_1 \equiv lim_{\sigma \to \infty} T = \left(\frac{s(1+\rho)}{(\gamma - s)} \right) \left[\frac{\gamma + (1-\gamma)s(1+\rho)}{\gamma} \right] > 0, \tag{16}$$

$$D_1 \equiv lim_{\sigma \to \infty} D = \left(\frac{s(1+\rho)}{(\gamma - s)} \right)^2 \left[\frac{(\gamma - 1)(\gamma - s)}{\gamma} \right] < 0, \tag{17}$$

$$lim_{\sigma \to 0} T = +\infty, \tag{18}$$

$$lim_{\sigma \to 0} D = +\infty. \tag{19}$$

From the expressions $(16)-(17)$, we first conclude that T_1 is always positive, while D_1 is always negative. Moreover, both T and D decrease in σ, and tend to $(T, D) = (+\infty, +\infty)$ when $\sigma \to 0$.

6. Geometrical Method

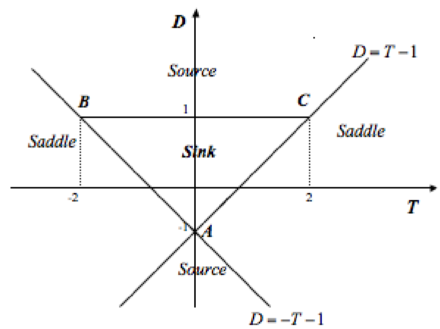
Our analysis of local dynamics uses the geometrical method developed in Grandmont et al. (1998). This method allows us to characterize the local stability properties around a neighborhood of the steady state. Therefore, we are able to analyze the occurrence of local indeterminacy and bifurcations in terms of relevant parameters (providing a direct economic interpretation). We consider γ as fixed throughout our analysis and study how (T, D) varies with changes in σ and s over their admissible ranges.

In this procedure, we evaluate the characteristic polynomial $P(\lambda) = \lambda^2 - \lambda T + D = 0$ at $\lambda = -1,\ 0,\ 1$. In the (T, D) space, we are allowed to perform the same analysis when representing three lines of interest. First, the line AC $(D = T - 1)$, where a local eigenvalue is equal to 1, i.e. $P(1) = 1 - T + D = 0$. Second, the line AB $(D = -T - 1)$, where a local eigenvalue is equal to -1, i.e. $P(-1) = 1 + T + D = 0$. Finally, the segment $[BC]$ ($D = 1$ and $\mid T \mid < 2$), where the two eigenvalues are complex conjugates with a unit modulus. Hence, when T and D are inside the triangle ABC the steady state is locally a sink (both eigenvalues have modulus lower than one) - asymptotically stable. The model consists in one predetermined variable, the capital stock, and one non-predetermined variable, the equity price. Accordingly, the steady state is locally indeterminate, i.e. there exists a continuum of equilibrium paths starting from the same initial capital stock that stay arbitrarily close to the steady state, if and only if the steady state is locally a sink[10]. If such holds true, then there

[10] Indeterminacy emerges if the number of eigenvalues with modulus lower than one is higher than the number of predetermined variables.

are infinitely many stochastic endogenous fluctuations in a neighborhood of the steady state[11]. In the other cases, the steady state is locally determinate. The steady state is thus a saddle when $\mid T \mid > \mid D+1 \mid$ and a source in all the remaining areas. The figure below characterizes the local stability properties in the (T, D) space, as described above.

Figure 1: Local dynamics in the (T, D) space



The half-line Δ

We start with the analysis of variations in the trace T and determinant D in the (T, D) space as one of the parameters of interest is made to vary continuously within its admissible range. This methodology allows us to characterize the local stability of the steady state, as well as the occurrence of local bifurcations. First, let us consider the locus of points $(T(\sigma),\, D(\sigma))$, obtained as the elasticity of capital-labour substitution is made to vary continuously within its admissible range $\left[0, +\infty\right)$. This locus of points $(T(\sigma),\, D(\sigma))$ describes a half-line Δ in the (T, D) space. The half-line Δ starts at $(T, D) = (+\infty, +\infty)$ and ends at (T_1, D_1), with T_1 and D_1 as defined in (16) and (17). As proved in Appendix $B.4$, the half-line Δ is equal to

$$D = \Delta(T) = \frac{s}{\gamma - s} T + \left(\frac{s}{\gamma - s}\right)^2 \left[(1+\rho)^2(\gamma - 1) - (1+\rho)\right]. \qquad (20)$$

Proof. See Appendix B.4 □

[11] Grandmont et al. (1998) prove the equivalence between a steady state that is stable (locally indeterminate) in the deterministic forward perfect foresight dynamics and the occurrence of infinitely many stochastic endogenous fluctuations in a neighborhood of the steady state when perfect foresight is absent.

The half-line Δ is therefore always positively slopped, with a slope higher than one if and only if $s > \gamma/2 \equiv s_1^*$. Moreover, the half-line Δ points downwards.

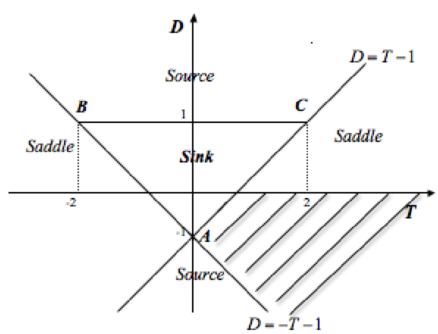
The half-line Δ_1

From $(16)-(17)$, it can be checked that both T_1 and D_1 vary in s. Hence, one needs to isolate the effect of a change in s over T_1 and D_1 in order to have a complete description of the behaviour of the half-line Δ when s changes within its admissible range. Accordingly, we then focus on the behavior of (T_1, D_1) as s varies in $(0, \gamma)$. From $(16)-(17)$, the locus of points $(T_1(s), D_1(s))$ obtained as s continuously increases from 0 to γ describes a half-line Δ_1, starting at $(T_1(0), D_1(0)) = ({}^1/\gamma^2, {}^{(\gamma-1)}/\gamma^2)$ and pointing downwards, such that $(T_1(\gamma), D_1(\gamma)) = (+\infty, -\infty)$. From $(T_1(0), D_1(0))$, we conclude that Δ_1 always starts to the right of C, with $T_1 > 1$ and $D_1 < 0$. Also, from (16) and (17) it can be checked that both D_1 and T_1 are nonlinear in s, making the computations needed to define explicitly a half-line Δ_1 very cumbersome. However, we are able to understand what is the admissible region for the half-line Δ_1 in the (T, D) space.

First, we compare the half-line Δ_1 with the line AB. We show that D_1 is above AB when $s(1+\rho)\gamma + \gamma(\gamma - s) > 0$, which is always verified. Second, we compare the half-line Δ_1 with the line AC. For simplicity's sake, we define $\varphi \equiv \frac{\Pi}{m\overline{p}} > 0$ and rewrite D_1 and T_1 in terms of φ and ρ. Then, we show that D_1 is below AC when $\frac{(\gamma-1)s}{\gamma\rho}\left[(1+\rho)\varphi + (1+\varphi)\right] < \varphi(1 + {}^s/\rho)$, where the left-hand side of the inequality is always negative and the right-hand side is always positive. Hence, the half-line Δ is always below AC. Summarizing, the half-line Δ_1 is always above the line AB and below the line AC, with $T_1 > 0$ and $D_1 < 0$. Figure 2 represents the admissible region for the half-line Δ_1 in the (T, D) space.

Proof. *See Appendix B.*5 □

Figure 2: Half-line Δ_1 in the space (T, D) lies in the shaded area



Local dynamics

In the previous sub-section we concluded that the half-line Δ_1 always verifies $T_1 > 0$ and $D_1 < 0$. Moreover, we proved that the half-line Δ_1 starts for $s = 0$ at $(T_1(0), D_1(0)) = (1/\gamma^2, (\gamma-1)/\gamma^2)$ and tends to $(T_1(\gamma), D_1(\gamma)) = (+\infty, -\infty)$ when s approaches γ from below. Finally, we showed that the half-line Δ_1 always lies between the lines AB and AC. We then conclude that indeterminacy may never emerge for $s \leq s_1^*$ (when the slope of the half-line Δ is lower than one), as the half-line Δ never crosses the region ABC. However, when the slope is higher than one it may be that the half-line Δ crosses the the line AC to the left of the point C. If such holds true, then the half line Δ also crosses the line $T = 2$ above $D = 1$. We therefore check if $\Delta(2)$ is above or below one. It can be shown that $\Delta(2)$ is higher than one if and only if

$$G(s, \gamma) \equiv -s^2(1+\rho)\left[(1+\rho)(1-\gamma)+1\right] - (\gamma - s)(\gamma - 3s) > 0 \qquad (21)$$

Proof. See Appendix B.6 □

where the first term is always negative. Notice that $G(s, \gamma)$ is always negative for $s \leq \gamma/3$, and thus (21) is never verified within this range of s. For $s > \gamma/3$ the two terms have an opposite sign. Hence, we need to understand which effect dominates in order to have a full description of the half-line Δ. For $\gamma/3 < s \leq s_1^*$ it must be that $G(s, \gamma) < 0$, as it is known that the half-line Δ cannot cross the line AC to the left of C given the admissible area for the half-line Δ_1 and a slope lower or equal to one (reductio ad absurdum argument). We then compute an analytical solution for $G(s, \gamma)$ conditional on $s \in (s_1^*, \gamma)$ and $\gamma \in (0, 1)$ and prove that the solution set to this problem is empty[12]. Therefore, the half-line Δ always intercepts the line $T = 2$ below C. We thus conclude that indeterminacy may not emerge.

Proposition 4. *The steady state is never locally a sink, and therefore local indeterminacy can not emerge, i.e. it is not possible that an infinite number of equilibrium trajectories starting from the same value of K_0 stay arbitrarily close to the steady state. The steady state is always locally determinate.*

In order to fully describe the local dynamics of our system we need to determine the value of the elasticity of capital-labour substitution at which the steady state undergoes a transcriptical bifurcation. The value of σ equal to σ_T at which a transcriptical bifurcation occurs verifies $D = T - 1$. After some computations

[12] An analytical solution were obtained using Wolfram Mathematica. The solution set for the inequality in (21) always requires values for $\gamma \in (1, +\infty)$. Accordingly, conditional on $\gamma \in (0, 1)$ the solution set to the inequality in (21) is empty.

it is shown that $\sigma_T \equiv \frac{(\gamma-2s)(1+\rho)^2}{2s(1+\rho)^2(\gamma-1)-(1+\rho)+(\gamma-s)}$, where the denominator is always negative. Accordingly, $s > s_1^*$ is a necessary condition for the steady state to undergo a transcriptical bifurcation (guarantees that $\sigma_T > 0$).

We may hence describe the dynamics of our system as in Proposition 5.

Proposition 5. *The local dynamics of the system* $(8)-(9)$ *may be described recurring to critical values* s_1^* *and* σ_T*, as follows:*

1. *When* $s \leq s_1^*$ *or* $s > s_1^*$ *and* $\sigma > \sigma_T$ *the steady state is locally a saddle. Therefore, there is a unique trajectory that for a given value* K_0 *for the capital stock converges asymptotically to the steady state.*
2. *At* $\sigma = \sigma_T$ *(given* $s > s_1^*$*) the steady state undergoes a transcriptical bifurcation, i.e. the two different fixed points cross each other, and exchange their stability properties.*
3. *When* $s > s_1^*$ *and* $\sigma < \sigma_T$ *the steady state is locally a source. Hence, for a given value* K_0 *there is a unique trajectory that diverges from the steady state.*

Proof. *See Appendix B.7* □

Figure 3 represents the half-line Δ in the (T, D) space for two different values of s and $\gamma = 0.95$. When $s = 0.25$, $s \leq s_1^*$ is verified, and therefore the steady state is always locally a saddle. When $s = 0.6$, $s > s_1^*$ is verified. Hence, for $\sigma > \sigma_T = 0.3788$ the steady state is locally a saddle, while for $\sigma < \sigma_T$ the steady state is locally a source.

Figure 3: The half-line Δ and Δ_1 in the (T, D) space with $\gamma = 0.95$

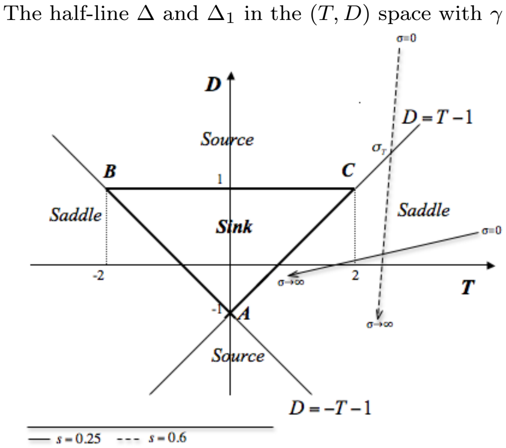
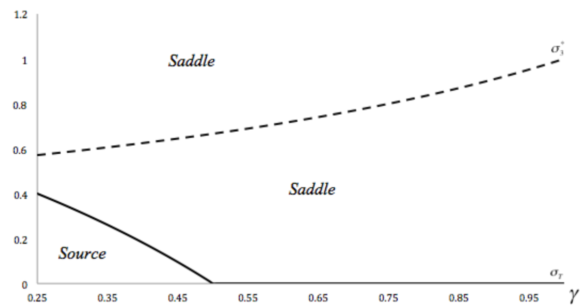


Figure 4 defines the local dynamics in the (s, σ) space for different values of γ and $s = 0.25$. The solid line splits the local dynamics in two regions - above the line the steady state state is locally a saddle, while below the line the steady state is locally a source. Therefore, the solid line represents the $max\,[0, \sigma_T]$ as a function of γ (for a value of s equal to 0.25). Furthermore, Figure 4 represents the value of $\sigma_3^* \equiv \frac{1}{2-\gamma}$ as a function of γ and allow us to compare the two critical values. Hence, it illustrates that $\sigma_3^* > \sigma_T$ (as proven analytically in the next section).

Figure 4: Local dynamics in the (s, σ) space for different values of γ and $s = 0.25$



7. Steady state welfare

We now restrict our attention to the comparison of steady state welfare. First, we define steady state welfare as measured by aggregate consumption at the steady state. Second, we compare the level of aggregate consumption in our economy with that of our benchmark, i.e. an economy with the same characteristics, but without a secondary market for shares. Our benchmark economy works as follows: young agents invest all their wage income earned at time t in productive capital. Productive capital is then rented out to one of the N firms (with total capital stock depreciation). In the next period, when retired, the young of period t earn both the real rental rate and total profits per firm. The total revenue stream received when old is then used to consume. Therefore, the budget constraints are respectively (for our economy and its benchmark)

$$K + \frac{A(1-\gamma)F(K,\bar{L})}{AF_K(K,\bar{L})-1} = g(K), \tag{22}$$

$$K = g(K), \tag{23}$$

with $g(K) \equiv w\overline{L}$ and $\frac{A(1-\gamma)F(K,\bar{L})}{AF_K(K,\bar{L})-1} > 0$. Also, $\varepsilon_{w\overline{L},K} \equiv \frac{\partial w\overline{L}}{\partial K}\frac{K}{w\overline{L}} > 0$ if and only if $\sigma < \frac{1}{(1-\gamma)} \equiv \sigma_1^*$. Moreover, we show that $\varepsilon_{w\overline{L},K} > 1$ if and only if $\sigma < \frac{s}{\gamma+s(1-\gamma)} \equiv \sigma_2^*$. Given $\gamma \in (0,1)$, σ_1^* is always larger than one. We also show that $0 \leq \sigma_2^* < 1$. Therefore, it is always ensured that $\sigma_2^* < \sigma_1^*$ holds true.

Proposition 6. *The slope of $g(K)$ may be defined recurring to the critical values σ_1^* and σ_2^*, as follows:*

(a) If $0 \leq \sigma \leq \sigma_2^*$ the slope of $g(K)$ is higher or equal to one;

(b) If $\sigma_2^* < \sigma \leq \sigma_1^*$ the slope of $g(K)$ is positive or flat, but always lower than one;

(c) If $\sigma_1^* \leq \sigma < +\infty$ the slope of $g(K)$ is negative.

Assumption 3. *We assume a CES production function, with an elasticity of substitution between capital and labour higher or equal to $\frac{1}{2-\gamma}$ (i.e. $\sigma > \frac{1}{2-\gamma} \equiv \sigma_3^*$).*

Using this assumption, we get that $\sigma > \sigma_2^*$ [13]. Therefore, either (b) or (c) hold true, and the slope of $g(K)$ is never higher than one. Also, as proven in Proposition 2, under $\sigma > \sigma_3^*$ the uniqueness of the steady state is guaranteed - the steady state is always a saddle and no transcriptical bifurcation occurs[14]. Accordingly, within this range for σ any steady state welfare analysis is fully informative.

Proof. See Appendix B.8 □

Now, consider K_1^* as the steady state value for the capital stock that satisfies (23). Also, let us define K_2^* as the steady state value for the capital stock that satisfies (22). Then, the relation between K_1^* and K_2^* may be defined along this line:

Proposition 7. *Under Assumption 3., it always holds that $K_2^* < K_1^*$.*

Proof. See Appendix B.9 □

[13] Under $\sigma > \sigma_3^*$, the condition $s > \frac{s\sigma}{1-\sigma(1-\gamma)} \equiv s^*$, equivalent to $\sigma < \sigma_2^*$, would require a value of s higher than γ to be verified.

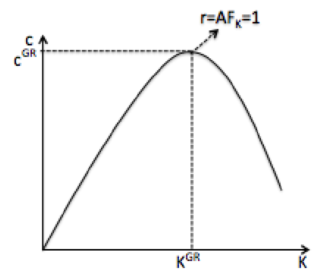
[14] An analytical solution for $\sigma_T > \sigma_3^*$ conditional on $s \in (s_1^*, \gamma)$ and $\gamma \in (0, 1)$ was computed using Wolfram Mathematica. The solution set to this problem was found to be empty. Therefore, $\sigma^T < \sigma_3^*$ always holds true. Accordingly, under Assumption 3 the steady state is always a saddle. In particular, under Assumption 3 the steady state never undergoes a transcriptical bifurcation, and the possibility of multiple steady states around a neighborhood of σ^T is ruled out. This proof is in line with the comparison between σ_T and σ_3^* in Figure 4.

We now focus on the relation between aggregate consumption at the steady state and capital stock. We start by writing the resource constraint for the two economies

$$AF(K_t, \bar{L}) = c_t + K_{t+1}.$$

Therefore, aggregate consumption at the steady state satisfies $c = AF(K, \bar{L}) - K$, with $\frac{\partial c}{\partial K} = AF_K(K, \bar{L}) - 1$ and $\frac{\partial^2 c}{\partial K^2} = AF_{KK}(K, \bar{L}) < 0$. The capital stock at the golden rule is thus such that verifies $r = AF_K(K, \bar{L}) = 1$. Figure 5 represents aggregate consumption as a function of aggregate capital stock.

Figure 5: Aggregate consumption



First, we focus on steady state consumption in the economy with a secondary stock market. From the arbitrage condition, the real rental rate at the steady state equals one plus the dividend yield, which is always positive (note that endogenous profits arise under decreasing returns to scale)[15]. Then, given $\frac{\partial r}{\partial K} > 0$, it always holds that $K_2^* < K^{GR}$. Hence, aggregate consumption is increasing in K in the economy with a secondary stock market. Accordingly, the steady state is for the economy with a secondary stock market said to be dynamic efficient.

Second, the real rental rate is for the benchmark economy equal to $\frac{s}{\gamma - s}$ at the steady state[16]. Therefore, the real rental rate is higher than one if and only if $s > \gamma/2 \equiv s_1^*$. Again, given $\frac{\partial r}{\partial K} > 0$, $K_1^* \leq K^{GR}$ if $s \geq s_1^*$ and $K_1^* > K^{GR}$ otherwise. Then, using this information and Proposition 7, we describe the

[15] From the arbitrage condition, $r_{t+1} = \frac{\frac{\Pi^N_{t+1}}{m} + p_{t+1}}{p_t}$. At the steady state, it holds that $r = 1 + \frac{\Pi}{mp}$, with $\frac{\Pi}{mp}$ positive under decreasing returns to scale. Therefore, it is always assured that $r = 1 + \frac{\Pi}{mp} > 1$.

[16] From the firms' problem we have that $r = A\overline{L}^{\gamma-1} f'(x) = \frac{f'(x)}{\left[\gamma f(x) - f'(x)x\right]} w = \frac{s}{\gamma - s}\frac{w}{x}$. Also, from the budget constraint, we have that $w\overline{L} = K \Leftrightarrow w = x$. Therefore, we may write the real rental rate at the steady state as $r = \frac{s}{\gamma - s}$.

behavior of aggregate consumption at the steady state in the economy without a secondary stock market.

Proposition 8. *Aggregate consumption at the steady state may be defined for the benchmark economy recurring to the critical value s_1^*, as follows:*

- $K_2^* < K_1^* < K^{GR}$ if $s_1^* < s < \gamma$. Accordingly, aggregate consumption is increasing in K, and therefore the steady state is dynamic efficient;
- $K_2^* < K_1^* = K^{GR}$ if $s = s_1^*$;
- $K_2^* < K^{GR} < K_1^*$ if $0 < s < s_1^*$. Accordingly, aggregate consumption is decreasing in K, and therefore the steady state is dynamic inefficient.

Now, consider $0 < s < s_1^*$, and thus dynamic inefficiency for the economy without a secondary stock market. Then, there is a value of s equal to s_2^*, with $s_2^* \in (0, s_1^*)$, such that aggregate consumption, and therefore steady state welfare, is equal in both economies, i.e. $c(K_1^*) = c(K_2^*)$ with $K_2^* < K^{GR} < K_1^*$. Finally, we compare steady state welfare in the two economies through Proposition 9.

Proposition 9. Steady state welfare in the two economies may be compared recurring to the critical values s_1^* and s_2^*, as follows:

- If $s \in [s_1^*, \gamma)$, then $c(K_2^*) < c(K_1^*)$ and the steady state is dynamic efficient for both economies. The introduction of a secondary stock market absorbs savings available for investment in productive capital, and therefore deteriorates steady state consumption.

- If $s \in [s_2^*, s_1^*)$, then $c(K_2^*) \leq c(K_1^*)$ and the steady state is dynamic inefficient for the benchmark economy (here designated mild dynamic inefficient). The introduction of a secondary stock market leads to under accumulation of capital. Under mild dynamic inefficiency, the decrease in the steady state value for the capital stock suffered with the introduction of a secondary stock market is too strong and deteriorates steady state welfare.

- If $s \in (0, s_2^*)$, then $c(K_2^*) > c(K_1^*)$ and the steady state is dynamic inefficient for the benchmark economy (here designated strong dynamic inefficient). The introduction of a secondary stock market absorbs savings available for investment in productive capital, correcting for the over accumulation of capital under dynamic inefficiency. Therefore, under strong dynamic inefficiency, i.e. $s \in (0, s_2^*)$, the introduction of a secondary stock market forces the economy to converge to a steady state closer to the golden rule path, so that future generations will benefit from higher consumption per capita, and hence be better-off. The intuition underlying the convergence to a steady state with higher aggregate consumption is that the introduction of a substitute asset (firms' shares) will force the substitution of savings in the form of productive capital investment with stock market investment.

Finally, it is important to mention that average values for the capital share of output within the OECD countries is around 1/3. Therefore, under empirically plausible values for the capital share of output, the introduction of a secondary stock market may be steady state welfare improving.

Part III

Concluding Remarks

We studied an OLG model with consumption in the second period of life, decreasing returns to scale in production (endogenous profits) and a secondary stock market. We have shown that locally indeterminate equilibria does not emerge. This statement holds true for all the admissible values of the capital share of output and the elasticity of capital-labour substitution, and is therefore quite robust. Under our assumptions, the steady state is either a saddle our a source. For empirically plausible values for the capital share of output and the elasticity of capital-labour substitution the steady state is locally a saddle. Accordingly, in most cases there is a unique deterministic equilibrium that stay in a neighborhood of the steady state, converging to it as $t \to +\infty$. However, if the capital share of output is sufficiently high and the elasticity of capital-labour substitution is sufficiently low the steady state is locally a source, meaning that there is a unique trajectory that diverges asymptotically from the steady state (explosive behavior).

On these findings, two considerations are worth to be mentioned. First, the introduction of a secondary stock market influences the dynamics of the model, as the steady state may become a source. Second, in this general equilibrium framework, there is no room for volatility generated by self-fulfilling changes in expectations. Hence, indeterminacy does not hold as a possible explanation for the excess volatility puzzle.

Furthermore, we show that steady state welfare may be lower when the trading of stocks is possible in the secondary market. However, under empirically plausible values for the capital share of output (i.e. under strong dynamic inefficiency), the introduction of a secondary stock market forces the economy to converge to a steady state with higher aggregate consumption. Note that the two assets are substitutes in this model. Therefore, the introduction of a secondary stock market contraries the over accumulation of capital verified under dynamic inefficiency, and may thus lead to an improvement in steady state welfare if the capital share of output is sufficiently low (or, in other words, if dynamic inefficiency is strong enough). We claim in this paper that this conclusion is fully informative given the uniqueness of the steady state (for values of $\sigma > \sigma_3^*$ the steady state is unique and always locally a saddle).

Accordingly, we argue that trading in the secondary stock market is a good instrument in controlling for dynamic inefficiency (and thus recommended for economies with strong over accumulation of capital). Moreover, it is an instrument without the cost of introducing dynamic instability.

Finally, for further research is advisable to consider the possibility of also current consumption, whereby saving decisions are influenced by the return on assets. Also, we propose to introduce complementarity between firms' shares and capital through the inclusion of a primary stock market (i.e. a market for new issues). This feature will help in correcting for the lack of dynamics on the firm side, one of the main limitations of macroeconomic dynamics models, and of our model in particular. We thus should study an OLG model with the same structure, but with firms deciding whether to issue shares to finance new projects. In this specification, the number of shares issued is a non predetermined variable, as it depends on the expected return on new projects. Additionally, we further plan to introduce dynamics on the firm side through the decision to use internal financing (i.e. the decision to retain or distribute profits). Simultaneously, and still in this model's setting, we propose to study how different fiscal policy rules, namely dividends and capital gains' taxation, influence the local dynamics of our system.

Part IV
Appendix

Appendix A. On the baseline parameter values

We start by computing the capital share in total income. From the Euler Condition we have that:

$\gamma AF(k,m) = k\frac{\partial AF(k,m)}{\partial k} + m\frac{\partial AF(k,m)}{\partial m} \Leftrightarrow$

$\Leftrightarrow \gamma = \frac{xf^{'}(x)}{f(x)} + \frac{\left[\gamma f(x)-f^{'}(x)x\right]}{f(x)} \Leftrightarrow \gamma = s + (\gamma - s), \quad with\, s = \frac{xf^{'}(x)}{f(x)}.$

Now, we focus on computing the elasticity of capital-labour substitution. The Marginal Rate of Technical Substitution equals $MRTS = \frac{\gamma f(x)-f^{'}(x)x}{f^{'}(x)}$. Furthermore, $\frac{\partial MRTS}{\partial x} = (\gamma-1) - \frac{f^{''}(x)\gamma f(x)}{(f^{'}(x))^2}$. Therefore,

$$\frac{1}{\sigma(x)} \equiv \frac{\partial MRTS}{\partial x}\frac{x}{MRTS} = \left\{\frac{(\gamma-1)f^{'}(x)x}{\gamma f(x)} - \frac{f^{''}(x)x}{f^{'}(x)}\right\}\frac{1}{1-\frac{f^{'}(x)x}{\gamma f(x)}} =$$

$$= \left\{\frac{(\gamma-1)}{\gamma}s - \frac{f^{''}(x)x}{f^{'}(x)}\right\}\frac{\gamma}{\gamma - s}.$$

For future reference, we also rearrange the formula above to obtain that

$$\frac{f^{''}(x)x}{f^{'}(x)} = \frac{1}{\gamma}\left\{(\gamma-1)s - \frac{(\gamma-s)}{\sigma}\right\}.$$

Appendix B. Steady state and local dynamics

Appendix B.1. Proof of Proposition 2

From $(10)-(11)$, a steady state (K,p) has to verify

(a) $A\bar{L}^{\gamma-1}\left[\gamma f(x)-f^{'}(x)x\right]-({}^{mp}/_{\overline{L}})=x$ and

(b) $1=A\left[\overline{L}^{\gamma-1}f^{'}(x)-\frac{(1-\gamma)\overline{L}^{\gamma}f(x)}{mp}\right]$

First, note that A is the unique solution of (b) for a steady state (K,p). Then, using (b) to substitute the value of A into (a) we obtain $Z(m)=1$, with

$$Z(m)\equiv\left[\frac{x+({}^{mp}/_{\overline{L}})}{\gamma f(x)-f^{'}(x)x}\right]\frac{1}{\overline{L}^{\gamma-1}}\left[\overline{L}^{\gamma-1}f^{'}(x)-\frac{(1-\gamma)\overline{L}^{\gamma}f(x)}{mp}\right].$$

Also, note that $Z(m)>0$ if $m>\frac{(1-\gamma)\overline{L}^{\gamma}f(x)}{\overline{L}^{\gamma-1}f^{'}(x)p}\equiv m^{0}$ and is continuously increasing in $m>m^{0}$, with $lim_{m\to+\infty}Z(m)=+\infty$, so that $Z(m)$ must cross once the value 1.

Consider now that A and m are such that (a) and (b) are satisfied for (K,p). Using (b), note that at the steady state p must satisfy$mp=\frac{A(1-\gamma)\overline{L}^{\gamma}f(x)}{A\overline{L}^{\gamma-1}f^{'}(x)-1}$. Then, let us write (a) as $g(K)-K=mp$, with $g(K)\equiv w\bar{L}=A\bar{L}^{\gamma}\left[\gamma f(x)-f^{'}(x)x\right]$. Moreover, as shown in Appendix $B.8$, $\varepsilon_{w\overline{L},K}\equiv\frac{\partial w\overline{L}}{\partial K}\frac{K}{w\overline{L}}=\frac{s(x)(\gamma-1)}{\gamma}+\frac{s(x)}{\gamma\sigma}$. Also, assuming a technology with a constant elasticity of substitution between capital and labor, we show in Appendix $B.8$ that $\sigma>\frac{1}{2-\gamma}\equiv\sigma_{3}^{*}$ is a sufficient condition to guarantee that the slope of $g(K)$ is lower than one. Accordingly, under $\sigma>\sigma_{3}^{*}$ it is assured that the slope of $g(K)-K$ is negative. Furthermore, we analyze the behavior of mp in K:

$$\frac{\partial mp}{\partial K}=\frac{\left\{A(1-\gamma)\overline{L}^{\gamma-1}f^{'}(x)\left[A\overline{L}^{\gamma-1}f^{'}(x)-1\right]+A^{2}\overline{L}^{2(\gamma-1)}(1-\gamma)f(x)f^{''}(x)\right\}}{\left[A\overline{L}^{\gamma-1}f^{'}(x)-1\right]^{2}}>0$$

Consequently, for a technology with a constant elasticity of substitution between capital and labour, under $\sigma>\sigma_{3}^{*}$ it is guaranteed that there is at most one value K that satisfies $g(K)-K=mp$. Given this value K there is a unique value p satisfying $p = \frac{A(1-\gamma)\overline{L}^{\gamma} f(x)}{m\left[A\overline{L}^{\gamma-1} f'(x)-1\right]}$, so that there is at most one steady state (K, p). Accordingly, as shown above, conditional on A and m ensuring the existence of a steady state (K, p), the latter is the unique steady state of the dynamic system when $\sigma > \sigma_3^*$.

Appendix B.2. Proof of Proposition 3

Log-linearizing (8) around the steady state:

$$\frac{dK_{t+1}}{K_{t+1}} = \frac{\bar{L}K_t}{K_{t+1}} \frac{\partial w_t}{\partial K_t}_{|s.s.} \frac{dK_t}{K_t} - \frac{p_t m}{K_{t+1}}_{|s.s} \frac{dp_t}{p_t} \Leftrightarrow$$

$$\Leftrightarrow \frac{dK_{t+1}}{K_{t+1}} = \frac{\bar{L}w}{K} \varepsilon_{w,K} \frac{dK_t}{K_t} - \frac{mp}{K} \frac{dp_t}{p_t} \Leftrightarrow$$

$$\Leftrightarrow \frac{dK_{t+1}}{K_{t+1}} = (1 + \frac{mp}{K})\varepsilon_{w,K} \frac{dK_t}{K_t} - \frac{mp}{K} \frac{dp_t}{p_t} \Leftrightarrow$$

$$\Leftrightarrow \frac{dK_{t+1}}{K_{t+1}} = (1+\rho)\varepsilon_{w,K} \frac{dK_t}{K_t} - \rho \frac{dp_t}{p_t}$$

with $\rho \equiv \frac{mp}{K}$ equal to the steady state value for the market-to-book ratio (or Tobin's q) and,

$$\varepsilon_{w,K} \equiv \frac{\partial w}{\partial K} \frac{K}{w} = \frac{(\gamma - 1) - \frac{f''(x)x}{f'(x)}}{\frac{\gamma f(x)}{x f'(x)} - 1} = \frac{s}{\gamma - s} \left\{ (\gamma - 1) - \frac{1}{\gamma} \left[(\gamma - 1)s - \frac{(\gamma - s)}{\sigma} \right] \right\} =$$

$$= \frac{s}{\gamma} \left[(\gamma - 1) + \frac{1}{\sigma} \right].$$

Log-linearizing (9) around the steady state:

$$\frac{dp_{t+1}}{p_{t+1}} = r_{t+1} \frac{p_t}{p_{t+1}}_{|s.s.} \frac{dp_t}{p_t} + \left\{ \frac{\partial r_{t+1}}{\partial K_{t+1}} K_{t+1} \frac{p_t}{p_{t+1}}_{|s.s} - \frac{\partial d_{t+1}}{\partial K_{t+1}} \frac{K_{t+1}}{p_{t+1}}_{|s.s} \right\} \frac{dK_{t+1}}{K_{t+1}} \Leftrightarrow$$

$$\Leftrightarrow \frac{dp_{t+1}}{p_{t+1}} = \left\{ r\varepsilon_{r,K} - \frac{\Pi}{mp}\varepsilon_{\Pi,K} \right\} \frac{dK_{t+1}}{K_{t+1}} + r\frac{dp_t}{p_t} \Leftrightarrow$$

$$\Leftrightarrow \frac{dp_{t+1}}{q_{t+1}} = \left\{ (1+\frac{\Pi}{mp})\varepsilon_{r,K} - \frac{\Pi}{mp}\varepsilon_{\Pi,K} \right\} \frac{dK_{t+1}}{K_{t+1}} + (1+\frac{\Pi}{mp})\frac{dp_t}{p_t} \Leftrightarrow$$

$$\Leftrightarrow \frac{dp_{t+1}}{p_{t+1}} = \left\{ (1+\varphi)\varepsilon_{r,K} - \varphi\varepsilon_{\Pi,K} \right\} (1+\rho)\varepsilon_{w,K}\frac{dK_t}{K_t}$$

$$+ \left\{ (1+\varphi) - \left[(1+\varphi)\varepsilon_{r,K} - \varphi\varepsilon_{\Pi,K} \right] \rho \right\} \frac{dp_t}{p_t},$$

with $\varphi \equiv \frac{\Pi}{mp}$ and,

$\varepsilon_{r,K} \equiv \frac{\partial r}{\partial K}\frac{K}{r} = \frac{f''(x)x}{f'(x)} = \frac{1}{\gamma}\left\{ (\gamma-1)s - \frac{(\gamma-s)}{\sigma} \right\},$

$\varepsilon_{\Pi,K} \equiv \frac{\partial \Pi}{\partial K}\frac{K}{\Pi} = \frac{f'(x)x}{f(x)} = s.$

Furthermore, at the steady state the following relations are verified

$\varphi = \frac{(1-\gamma)(1+\rho)}{(\gamma-s)(1-\alpha)\rho}$ and $1+\varphi = \frac{s(1+\rho)}{(\gamma-s)(1-\alpha)}.$

After some straightforward computations, we obtain that the Trace and Determinant of the Jacobian matrix are given by

$T = (1+\rho)\varepsilon_{w,K} - \rho\left[\frac{s(1+\rho)}{(\gamma-s)}\varepsilon_{r,K} - \frac{(1-\gamma)(1+\rho)}{(\gamma-s)\rho}\varepsilon_{\Pi,K} \right] + \frac{s(1+\rho)}{(\gamma-s)},$

$D = (1+\rho)\varepsilon_{w,K}\frac{s(1+\rho)}{(\gamma-s)}.$

Then, we re-arrange the Determinant and Trace of the Jacobian matrix in order to have it represented in the form:

$T = \frac{s(1+\rho)}{\gamma(\gamma-s)}\left[\gamma + (1-\gamma)s(1+\rho) \right] + \frac{1}{\sigma}\left[\frac{s(1+\rho)^2}{\gamma} \right],$

$D = \frac{\left[(1+\rho)s(\gamma-1) + \frac{(1+\rho)s}{\sigma} \right]}{\gamma}\frac{s(1+\rho)}{(\gamma-s)}.$

We may now easily evaluate T and D at the boundary values for σ, obtaining the start and end values for the Trace and Determinant.

$T_1 \equiv lim_{\sigma \to +\infty} T = \left(\frac{s(1+\rho)}{(\gamma - s)} \right) \left[\frac{\gamma + (1-\gamma)s(1+\rho)}{\gamma} \right],$

$D_1 \equiv lim_{\sigma \to +\infty} D = \left(\frac{s(1+\rho)}{(\gamma - s)} \right)^2 \left[\frac{(\gamma - 1)(\gamma - s)}{\gamma} \right],$

$T_2 \equiv lim_{\sigma \to 0} T = +\infty,$

$D_2 \equiv lim_{\sigma \to 0} D = +\infty.$

Appendix B.3. Proof value for ρ

At the steady state, the dividend price ratio must verify:

$\frac{\Pi}{mp} = \frac{Am^{\gamma}(1-\gamma)f(x)}{mp} = \frac{Am(1-\gamma)f(x)}{Am^{\gamma-1}\left[\gamma f(x) - f'(x)x\right]} \frac{w}{mp} = \frac{(1-\gamma)}{(\gamma - s)} \frac{wm}{mp} = \frac{(1-\gamma)}{(\gamma - s)} \frac{K+mp}{mp} =$
$= \frac{(1-\gamma)(1+\rho)}{(\gamma - s)\rho}.$

Also, at the steady state the real rental rate must verify:

$r = \frac{Am^{\gamma-1}f'(x)x}{Am^{\gamma-1}\left[\gamma f(x) - f'(x)x\right]} \frac{w}{x} = \frac{s}{\frac{\gamma}{s} - 1} \frac{w}{x} = \frac{s}{\gamma - s} \frac{K+p}{K} = \frac{s(1+\rho)}{\gamma - s}.$

Consequently, from the arbitrage condition in (2) it must be that

$1 + \frac{\Pi}{mp} = r \Leftrightarrow 1 + \frac{(1-\gamma)(1+\rho)}{(\gamma - s)\rho} = \frac{s(1+\rho)}{\gamma - s} \Leftrightarrow$
$\Leftrightarrow (1+\rho)\rho s = (\gamma - s)\rho + (1-\gamma)(1+\rho) \Leftrightarrow$
$\Leftrightarrow \rho s + \rho^2 s = (\gamma - s)\rho + (1-\gamma) + (1-\gamma)\rho \Leftrightarrow$
$\Leftrightarrow s\rho^2 - \rho\left[1 - 2s\right] - (1-\gamma) = 0 \Leftrightarrow$
$\Leftrightarrow \rho = \frac{[1-2s] \pm \sqrt{[1-2s]^2 + 4s(1-\gamma)}}{2s}.$

However, $\rho = \frac{[1-2s] + \sqrt{[1-2s]^2 + 4s(1-\gamma)}}{2s}$ is the only root that verifies $\rho > 0$, and hence the binding constraint.

Appendix B.4. Proof of half-line Δ

Let the Determinant and the Trace of the Jacobian matrix be $D = \frac{Z_1 + Z_2 \frac{1}{\sigma}}{Z_3 + Z_4 \frac{1}{\sigma}}$ and $T = \frac{Z_5 + Z_6 \frac{1}{\sigma}}{Z_3 + Z_4 \frac{1}{\sigma}}$. Then, $\frac{1}{\sigma} = \frac{Z_3 T - Z_5}{Z_6 - Z_4 T}$. We may then define the locus of points $(T(\sigma), D(\sigma))$ obtained as σ varies from 0 to $+\infty$ as a half-line $\triangle$, such that

$$D = \Delta(T) = \frac{(Z_1 Z_6 - Z_2 Z_5) + (Z_2 Z_3 - Z_4 Z_1)T}{(Z_3 Z_6 - Z_4 Z_5)}.$$

From the previous section we may write:

$Z_1 = \frac{s(1+\rho)}{(\gamma-s)}(1+\rho)s(\gamma-1)$, $Z_2 = \frac{s(1+\rho)}{(\gamma-s)}(1+\rho)s$, $Z_3 = \gamma$,
$Z_4 = 0$, $Z_5 = \frac{s(1+\rho)}{(\gamma-s)}\left[\gamma + s(1+\rho)(1-\gamma)\right]$, $Z_6 = s(1+\rho)^2$.

Hence,

$$Z_1 Z_6 - Z_2 Z_5 = \left(\frac{s(1+\rho)}{(\gamma-s)}\right)^2 (1+\rho)s\gamma\left[(1+\rho)(\gamma-1)-1\right],$$

$$Z_2 Z_3 - Z_4 Z_1 = \left(\frac{s(1+\rho)}{(\gamma-s)}\right)(1+\rho)s\gamma,$$

$$Z_3 Z_6 - Z_4 Z_5 = \left(\frac{s(1+\rho)}{(\gamma-s)}\right)(\gamma-s)\gamma(1+\rho).$$

Therefore,

$$D = \Delta(T) = \frac{s}{\gamma-s}T + \left(\frac{s}{\gamma-s}\right)^2 (1+\rho)\left[(1+\rho)(\gamma-1)-1\right].$$

Appendix B.5. Proof of half-line Δ_1

First, notice that at the starting point $D_1 < 0$ and $T_1 > 0$. Second, lets compare the half-line Δ_1 with the line AB.

$D_1 > -T_1 - 1 \Leftrightarrow \frac{s^2(1+\rho)^2}{(\gamma-s)^2}\left[\frac{(\gamma-1)(\gamma-s)}{\gamma}\right] > -\frac{s(1+\rho)}{(\gamma-s)}\left[\frac{\gamma+(1-\gamma)s(1+\rho)}{\gamma}\right] - 1 \Leftrightarrow s(1+\rho)\gamma + \gamma(\gamma-s) > 0$,

which is always verified for all the admissible parameter values. Hence, the half-line Δ_1 is always above the line AB. Finally, let us compare the half-line Δ_1 with the line AC. For simplicity's sake, let us again denote $\varphi \equiv \frac{\Pi}{mp}$ and represent D_1 and T_1 in terms of ρ and φ.

$$D_1 = \frac{s(\gamma-1)(1+\rho)(1+\varphi)}{\gamma\rho}$$

$$T_1 = \frac{s(\gamma-1)(1+\rho)}{\gamma\rho} + (1+\varphi) - \frac{(\gamma-1)s(1+\varphi)}{\gamma\rho} + \frac{\varphi s}{\rho}$$

Then,

$$D_1 < T_1 - 1 \Leftrightarrow \frac{s(\gamma-1)(1+\rho)(1+\varphi)}{\gamma\rho} < \frac{s(\gamma-1)(1+\rho)}{\gamma\rho} + \varphi - \frac{(\gamma-1)s(1+\varphi)}{\gamma\rho} + \frac{\varphi s}{\rho} \Leftrightarrow$$
$$\Leftrightarrow \frac{s(\gamma-1)}{\gamma\rho}\left[(1+\rho)\varphi + (1+\varphi)\right] < \varphi(1+\tfrac{s}{\rho}),$$

where the left-hand side of the equation is always negative, while the right-hand side of the equation is always positive. Therefore, the relation is always verified, and the half-line Δ_1 is always below the line AC. We thus conclude that the half-line Δ_1 is such that verifies $D_1 < T_1 - 1$, $D_1 > -T_1 - 1$, $D_1 < 0$ and $T_1 > 0$.

Appendix B.6. Proof $\Delta(2) \lesseqgtr 1$

First, we start by evaluating the half-line Δ at $T = 2$,

$$\Delta(2) = \frac{2s}{\gamma-s} + \left(\frac{s}{\gamma-s}\right)^2 \left[(1+\rho)^2(\gamma-1) - (1+\rho)\right] =$$
$$= \frac{2s(\gamma-s)}{(\gamma-s)^2} + \left(\frac{s}{\gamma-s}\right)^2 \left[(1+\rho)^2(\gamma-1) - (1+\rho)\right]$$

Then, we check when is $\Delta(2) > 1 \Leftrightarrow$

$$\Leftrightarrow 2s(\gamma-s) + s^2\left[(1+\rho)^2(\gamma-1) - (1+\rho)\right] > (\gamma-s)^2 \Leftrightarrow$$

$$\Leftrightarrow G(s,\gamma) \equiv -s^2\left[(1+\rho)^2(1-\gamma) + (1+\rho)\right] - (\gamma-s)(\gamma-3s) > 0$$

where the first term is always negative. If $s \leq \gamma/3$, then $G(s,\gamma) > 0$ never holds and $\Delta(2) < 1$. However, this is not a valuable information, as it was already known that for $s \leq s_1^*$ (slope of half-line $\Delta \leq 1$), $\Delta(2) > 1$ cannot hold - given the admissible region for the half-line Δ_1 (reductio ad absurdum argument). Then, we are only uncertain about $\Delta(2) \gtrless 1$ for $s > s_1^*$. When $s > s_1^*$ the two terms in $G(s,\gamma)$ have an opposite sign, and an analytical solution to $G(s,\gamma)$ is very cumbersome to compute. However, we used Wolfram Mathematica to obtain the solution set to the problem $G(s,\gamma) > 0$ conditional on $s \in (s_1^*, \gamma)$ and $\gamma \in (0,1)$. The solution set to this problem was found to be empty, and therefore $G(s,\gamma) < 0$ always holds within $s \in (s_1^*, \gamma)$. Hence, it can be analytically proved that $\Delta(2) < 1$ is always verified within the admissible range for s.

Appendix B.7. Proof value for σ_T

From the proofs above, we know that the half-line Δ starts above the line AB and below the line AC, with $D_1 < 0$ and $T_1 > 0$. Moreover, it is known that the slope of the half-line Δ is higher than one if $s > s_1^*$, and that the half-line Δ never intercepts the line AC to the left of C. Therefore, for $s \leq s_1^*$ the steady state is always a saddle. However, for $s > s_1^*$ the steady state may be either a saddle or a source. We then define the local dynamics of our system by referring to the critical value of σ equal to σ_T, at which the steady state undergoes a transcriptical bifurcation.

The steady state undergoes a transcriptical bifurcation when $\Delta(T) = T - 1$,

$$\Delta(T) = T - 1 \Leftrightarrow \frac{(1+\rho)^2 s^2(\gamma-1)}{\gamma(\gamma-s)} + \frac{(1+\rho)^2 s^2}{\gamma(\gamma-s)\sigma} = \frac{s\gamma(1+\rho)}{\gamma(\gamma-s)} - \frac{(1+\rho)^2 s^2(\gamma-1)}{\gamma(\gamma-s)} + \frac{(1+\rho)^2 s}{\gamma\sigma} - 1 \Leftrightarrow \sigma = \frac{(\gamma-2s)(1+\rho)^2}{2s(1+\rho)^2(\gamma-1)-(1+\rho)+(\gamma-s)} \equiv \sigma_T$$

Also, the denominator is always negative. Therefore, it holds that a necessary condition to assure the existence of a bifurcation (meaning $\sigma_T > 0$) is that $s > s_1^*$.

Appendix B.8. Slope of $g(K)$

We start by analyzing how does $g(K) \equiv w\overline{L} = A\overline{L}^{\gamma}\left[\gamma f(x) - f^{'}(x)x\right]$ behave in K

$$\frac{\partial w\overline{L}}{\partial K} = A\overline{L}^{\gamma-1}\left[(\gamma-1)\,f^{'}(x) - f^{''}(x)x\right] = \frac{\left[(\gamma-1)f^{'}(x) - f^{''}(x)x\right]}{\left[\gamma f(x) - f^{'}(x)x\right]}w =$$

$$= \frac{\left[(\gamma-1) - \frac{f^{''}(x)x}{f^{'}(x)}\right]}{\left[\frac{\gamma f(x)}{xf^{'}(x)} - 1\right]}\frac{w}{x} = \left[(\gamma-1) - \frac{f^{''}(x)x}{f^{'}(x)}\right]\frac{s}{\gamma-s}\frac{w}{x} =$$

$$= \left[\frac{s(\gamma-1)}{\gamma} + \frac{s}{\gamma\sigma}\right]\frac{w}{x}$$

Therefore,

$$\varepsilon_{w\overline{L},K} \equiv \frac{\partial w\overline{L}}{\partial K}\frac{K}{w\overline{L}} = \frac{s(\gamma-1)}{\gamma} + \frac{s}{\gamma\sigma}$$

Now, assuming a technology with a constant elasticity of substitution between capital and labor, we compute two critical values. First, $\varepsilon_{w\overline{L},K} \geq 0$ when $\sigma \leq \sigma_1^* \equiv \frac{1}{1-\gamma}$. Second, we have that $\varepsilon_{w\overline{L},K} \geq 1$ when $\sigma < \frac{s}{\gamma+s(1-\gamma)} \equiv \sigma_2^*$. Now, note that under decreasing returns to scale $\sigma_1^* > 1$ always holds true. Moreover, it is always verified that $0 \leq \sigma_2^* < 1$. Hence, $\sigma_2^* < \sigma_1^*$ is always guaranteed.

Therefore, the slope of $g(K)$ may be defined as follows:

1. Positive and higher or equal to one when $0 \leq \sigma \leq \sigma_2^*$;
2. Positive or flat, but lower than one, when $\sigma_2^* < \sigma \leq \sigma_1^*$;
3. Negative when $\sigma_1^* < \sigma < +\infty$.

Now, let us write $\sigma \leq \sigma_2^*$ in terms of s

$\sigma \leq \frac{s}{\gamma+s(1-\gamma)} \equiv \sigma_2^* \Leftrightarrow \sigma\gamma \leq s\left[1-\sigma(1-\gamma)\right]$. Then, $1-\sigma(1-\gamma) > 0$ when $\sigma < \frac{1}{1-\gamma} \equiv \sigma_1^*$, which is always verified under $\sigma \leq \sigma_2^*$.

Also, note that $\sigma \leq \sigma_2^*$ is equivalent to $s \geq \frac{s\sigma}{1-\sigma(1-\gamma)} \equiv s^*$. However, $s \geq s^*$ may only hold if $s^* \leq \gamma$. Accordingly,

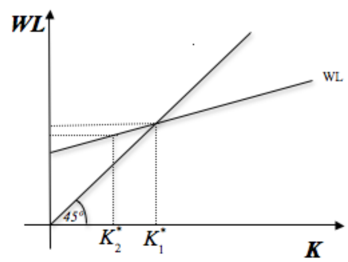
$$\frac{s\sigma}{1-\sigma(1-\gamma)} \equiv s^* \leq \gamma \Leftrightarrow \sigma \leq \frac{1}{2-\gamma} \equiv \sigma_3^*.$$

Taking this information into consideration, we will assume a CES production function with $\sigma > \sigma_3^*$ (e.g. Cobb-Douglas, linear etc.). Under $\sigma \in (\sigma_3^*, +\infty)$, the uniqueness of the steady state is always guaranteed (See Appendix A.1). Also, under our assumption for the production function it always holds that the slope of $g(K)$ is lower than one.

Appendix B.9. Comparing K_1^* and K_2^*

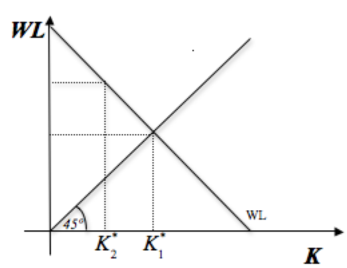
We then perform a local analysis to compare the steady state value for the capital stock in two economies. The following analysis will be performed under Assumption 3., i.e. $\sigma \in (\sigma_3^*, +\infty)$. First, let us focus on the value for the capital stock equal to K_1^* that satisfies $g(K) = K$ under $\sigma \in (\sigma_3^*, \sigma_1^*]$. Then, the value K_2^* that satisfies $g(K) = K + mp$ (with $mp > 0$) must be lower than K_1^*. Furthermore, it always holds that $\frac{\partial mp}{\partial K} > 0$. The figure below plots $g(K) \equiv w\overline{L}$ against a 45º line and the correspondent value for K_1^*. From Figure B.6 it can be confirmed that the steady state value K_2^* that is obtained when the 45º line moves upwards is always lower than K_1^*.

Figure B.6: Comparing K_1^* and K_2^* under $\sigma \in [1, \sigma_1^*]$



Second, consider the value for the capital stock equal to K_1^* that satisfies $g(K) = K$ under $\sigma \in (\sigma_1^*, +\infty)$. Again, we conclude that the value K_2^* that satisfies $g(K) = K + mp$ (with $mp > 0$) must be lower than K_1^*. We apply the same procedure and plot $g(K) \equiv w\overline{L}$ against a 45º line and the correspondent value for K_1^*. From Figure B.7 we check that the steady state value K_2^* that is obtained when the 45º line moves upwards is always lower than K_1^*.

Figure B.7: Comparing K_1^* and K_2^* under $\sigma \in (\sigma_1^*, +\infty)$



Part V
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